Submitted to the Annals of Statistics

SUPPLEMENT TO "FOURIER ANALYSIS OF STATIONARY TIME SERIES IN FUNCTION SPACE"

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Any reference to a formula or statement contained in the present document starts with a capital letter (eg 'A.1'). Any reference not of this form is to the main paper Panaretos & Tavakoli (2012).

APPENDIX A: THE SPECTRAL DENSITY OPERATOR

The autocovariance operators encode all the second-order dynamical properties of the series and are typically the main focus of functional time series analysis. Since we wish to formulate a framework for a frequency domain analysis of the series $\{X_t\}$, we need to determine a suitable notion of Fourier transform of these operators. This we call the *spectral density operator* of $\{X_t\}$, defined rigorously in Proposition A.1 below.

The following summarizes the basic property of the spectral density operator. Results of a similar flavour related to Fourier transforms between general Hilbert spaces can be traced back to, for example, Kolmogorov (1978); we give here the precise versions that we will be requiring, for completeness, since those results do not readily apply in our setting.

PROPOSITION A.1. *Suppose $p = 2$ or $p = \infty$, and consider the following conditions:*

I(p). *The autocovariance kernels satisfy $\sum_{t\in\mathbb{Z}} \|r_t\|_p < \infty$,*
II. *The autocovariance operators satisfy $\sum_{t\in\mathbb{Z}} |||\mathscr{R}_t|||_1 < \infty$,*

*where $|||\mathscr{R}_t|||_1$ is the nuclear norm or Schatten 1-norm (see Paragraph F.1.1). Then, under **I(p)**, for any $\omega \in \mathbb{R}$, the following series converges in $\|\cdot\|_p$:*

$$f_\omega(\cdot,\cdot) = \frac{1}{2\pi}\sum_{t\in\mathbb{Z}} \exp(-\mathrm{i}\omega t) r_t(\cdot,\cdot). \tag{A.1}$$

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*Research Supported by a European Research Council Starting Grant Award.

AMS 2000 subject classifications: Primary 62M10; secondary 62M15, 60G10

Keywords and phrases: cumulants, discrete Fourier transform, functional data analysis, functional time series, mixing, periodogram kernel, spectral density operator

We call the limiting kernel f_ω the spectral density kernel *at frequency ω. It is uniformly bounded and also uniformly continuous in ω with respect to $\|\cdot\|_p$, meaning given $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$|\omega_1 - \omega_2| < \delta \implies \left\|f_{\omega_1} - f_{\omega_2}\right\|_p < \varepsilon.$$

The spectral density operator $\mathscr{F}_\omega$, the operator induced by the spectral density kernel through right-integration, is self-adjoint and non-negative definite for all $\omega \in \mathbb{R}$. Furthermore, the following inversion formula holds in $\|\cdot\|_p$:

$$\int_0^{2\pi} f_\alpha(\tau,\sigma) e^{\mathrm{i}t\alpha} \mathrm{d}\alpha = r_t(\tau,\sigma), \forall t, \tau, \sigma. \tag{A.2}$$

*Under only **II**, we have*

$$\mathscr{F}_\omega = \sum_{t\in\mathbb{Z}} e^{-\mathrm{i}\omega t} \mathscr{R}_t, \tag{A.3}$$

where the convergence holds in nuclear norm. In particular, the spectral density operators are nuclear, and $|||\mathscr{F}_\omega|||_1 \leq \sum_t |||\mathscr{R}_t|||_1 < \infty$.

Proof of Proposition A.1. The convergence of (A.3) in $\|\cdot\|_p$ and the uniform boundedness of the spectral density kernel follows from the triangle inequality. Using the property $r_t(\tau,\sigma) = \left[r_{-t}(\sigma,\tau)\right]^{\mathsf{T}}$, we obtain $f_\omega(\tau,\sigma) = \left[f_\omega(\sigma,\tau)\right]^\dagger$, so that the spectral density operator $\mathscr{F}_\omega$ is self-adjoint. For uniform continuity, notice that

$$\begin{aligned}\left\|f_{\omega_1} - f_{\omega_2}\right\|_p &\leq \sum_{t\in\mathbb{Z}} \left|e^{-\mathrm{i}t\omega_1} - e^{-\mathrm{i}t\omega_2}\right| \|r_t\|_p \\ &\leq C \sum_{|t|\leq N} \left|e^{-\mathrm{i}t\omega_1} - e^{-\mathrm{i}t\omega_2}\right| + 2 \sum_{|t|>N} \|r_t\|_p,\end{aligned} \tag{A.4}$$

where $C = \max_{t\in\mathbb{Z}} \|r_t\|_p$. Fixing $\varepsilon > 0$, since I(p) holds, we can choose $N = N(\varepsilon) > 0$ such that the righthand sum of (A.4) is smaller than $\varepsilon/2$. Now since for each t, the function $\omega \mapsto e^{-\mathrm{i}t\omega}$ is uniformly continuous, we can choose a $\delta = \delta(N, \varepsilon) > 0$ such that the lefthand side of (A.4) is smaller than $\varepsilon/2$. Since $\delta = \delta(N(\varepsilon), \varepsilon) = \delta(\varepsilon)$, uniform continuity follows.

For the non-negativity, we need to show that $\langle \mathscr{F}_\omega \varphi, \varphi \rangle \geq 0$, for any $\varphi \in L^2([0,1], \mathbb{C})$. Our approach is inspired by Brillinger (2001). Let

$$\varphi^{(T)}(\tau) = \int_0^1 p_\omega^{(T)}(\tau,\sigma)\varphi(\sigma) d\sigma,$$

where

$$p_\omega^{(T)}(\tau,\sigma) = T^{-1}\sum_{t,s=0}^{T-1} e^{-\mathrm{i}\omega(t-s)} X_t(\tau)X_s(\sigma).$$

The element $\varphi^{(T)}$ is a random element of $L^2([0,1],\mathbb{C})$ since $\mathbb{E}\left\|\varphi^{(T)}\right\|^2 < \infty$. Hence, its expectation exists and Lemma F.3 yields

$$\mathbb{E}\varphi^{(T)} = T^{-1}\left(A_0 + \cdots + A_{T-1}\right)\varphi,$$

where $A_T = \sum_{t=-T}^{T} e^{-\mathrm{i}\omega t}\mathscr{R}_t$. Since $A_T \to \mathscr{F}_\omega$ in $|||\cdot|||_2$, the convergence also holds in operator norm $|||\cdot|||_\infty$. Therefore, the Cesàro-sum

$$T^{-1}(A_0 + \cdots + A_{T-1})$$

also converges to $\mathscr{F}_\omega$ in operator norm and $\mathbb{E}\varphi^{(T)} \to \mathscr{F}_\omega\varphi$ in the norm of $L^2([0,1],\mathbb{C})$. Since

$$\left\langle \varphi^{(T)},\varphi\right\rangle = \int_0^1 \left|\left((2\pi T)^{-1/2}\sum_{t=0}^{T-1} X_t(\tau)\exp(-\mathrm{i}\omega t)\right)\varphi(\tau)\right|^2 d\tau \geq 0,$$

we obtain

$$\langle \mathscr{F}_\omega\varphi,\varphi\rangle = \left\langle \lim_T \mathbb{E}\varphi^{(T)},\varphi\right\rangle = \lim_T \mathbb{E}\left\langle\varphi^{(T)},\varphi\right\rangle \geq 0.$$

The inversion formula follows from permuting the sum and integral, using the dominated convergence Theorem. It is done directly in the case $p=\infty$; in the case $p=2$, it is done by proving equality of the projections

$$\left\langle \int_{-\pi}^{\pi} f_\omega e^{it\omega}d\omega,\varphi\right\rangle = \langle r_t,\varphi\rangle$$

for all $\varphi \in L^2([0,1]^2,\mathbb{C})$, where we abuse from notation by writing $\langle\varphi,g\rangle = \int_{[0,1]^2}\varphi(x)\overline{g(x)}dx$ for $\varphi, g \in L^2([0,1]^2,\mathbb{C})$.

If **II** holds, then $I(2)$ holds and the spectral density kernel is defined in an L^2 sense. Equation (A.3) then follows from triangle inequality. □

APPENDIX B: HIGHER-ORDER CUMULANT SPECTRA

Provided

$$\text{(B.1)} \qquad \sum_{t_1,\ldots,t_{k-1}=-\infty}^{\infty} \|\mathrm{cum}\left(X_{t_1},\ldots,X_{t_{k-1}},X_0\right)\|_2 < \infty, \quad \forall\, k \in \mathbb{N}$$

holds true, we may also introduce the notion of a higher order spectral density, the *cumulant spectral density of order k*, defined in L^2:

$$f_{\omega_1,\ldots,\omega_{k-1}}(\tau_1,\ldots,\tau_k)$$
$$= \frac{1}{(2\pi)^{k-1}} \sum_{t_1,\ldots,t_{k-1}=-\infty}^{\infty} \exp\Big(-\mathbb{i}\sum_{j=1}^{k-1}\omega_j t_j\Big) \mathrm{cum}\left(X_{t_1}(\tau_1),\ldots,X_{t_{k-1}}(\tau_{k-1}),X_0(\tau_k)\right).$$

In shorthand, we will write

$$f_{\omega_1,\ldots,\omega_{k-1}} = \frac{1}{(2\pi)^{k-1}} \sum_{t_1,\ldots,t_{k-1}=-\infty}^{\infty} \exp\Big(-\mathbb{i}\sum_{j=1}^{k-1}\omega_j t_j\Big) \mathrm{cum}\left(X_{t_1},\ldots,X_{t_{k-1}},X_0\right).$$

We note that this density will be bounded. In fact, the convergence of the series defining the higher order density can be described explicitly as follows:

LEMMA B.1. *We have*

$$f_{\omega_1,\ldots,\omega_{k-1}} = \frac{1}{(2\pi)^{k-1}} \sum_{t_1,\ldots,t_{k-1}=-(T-1)}^{T-1} \exp\Big(-\mathbb{i}\sum_{j=1}^{k-1}\omega_j t_j\Big) \mathrm{cum}\left(X_{t_1},\ldots,X_{t_{k-1}},X_0\right) + \varepsilon_T,$$

where the equality is in L^2. The error term is uniform in ω, and satisfies $\varepsilon_T \sim o(1)$ as $T\to\infty$ if (B.1) holds. Under the stronger condition

$$\sum_{t_1,\ldots,t_{k-1}=-\infty}^{\infty} (1+|t_j|)\left\|\mathrm{cum}\left(X_{t_1},\ldots,X_{t_{k-1}},X_0\right)\right\|_2 < \infty, \quad \textit{for } j=1,\ldots,k-1,$$

we have $\varepsilon_T \sim o(T^{-1})$, as $T\to\infty$.

PROOF. Direct consideration of the expression for $f_{\omega_1,\ldots,\omega_{k-1}}$ yields that $|\varepsilon_T| \le \frac{1}{(2\pi)^{k-1}}\sum_{\nu=1}^{k-1}\Big[\sum_{|t_\nu|\ge T}\sum_{\substack{u\neq\nu\\ t_u\in\mathbb{Z}}}\left\|\mathrm{cum}\left(X_{t_1},\ldots,X_{t_{k-1}},X_0\right)\right\|_2\Big]$ and the estimates of the error follow directly. In particular, the error is independent of the ω's. □

With regards to higher order moments, we may establish an asymptotic representation of the cumulant kernel of the functional discrete Fourier transform in terms of the cumulant spectral density of the same order:

THEOREM B.2. *Let $\Delta^{(T)}(\omega)=\sum_{t=0}^{T-1}\exp(-\mathbb{i}\omega t)$. We have*

$$T^{k/2}\,\mathrm{cum}\left(\widetilde{X}^{(T)}_{\omega_1}(\tau_1),\ldots,\widetilde{X}^{(T)}_{\omega_k}(\tau_k)\right)$$
$$= (2\pi)^{k/2-1}\Delta^{(T)}(\omega_1+\cdots+\omega_k) f_{\omega_1,\ldots,\omega_{k-1}}(\tau_1,\ldots,\tau_k) + \varepsilon_T, \quad \textit{in } L^2,$$

where the error term is uniform in ω. In particular, $\varepsilon_T \sim o(T)$ if

$$\sum_{t_1,\ldots,t_{k-1}=-\infty}^{\infty} \left\| \mathrm{cum}\left(X_{t_1},\ldots,X_{t_{k-1}},X_0\right)\right\|_2 < \infty,$$

and $\varepsilon_T \sim O(1)$ if

$$\sum_{t_1,\ldots,t_{k-1}=-\infty}^{\infty} (1+|t_j|) \left\| \mathrm{cum}\left(X_{t_1},\ldots,X_{t_{k-1}},X_0\right)\right\|_2 < \infty, \quad \text{for } j=1,\ldots,k-1.$$

Notice that in the case $k=1$, $f(\tau)=\mu(\tau)$.

PROOF. We have

$$\begin{aligned}
&\mathrm{cum}\left(\widetilde{X}^{(T)}_{\omega_1}(\tau_1),\ldots,\widetilde{X}^{(T)}_{\omega_k}(\tau_k)\right)\\
&= (2\pi T)^{-k/2} \sum_{t_1,\ldots,t_k=0}^{T-1} e^{-\mathrm{i}\sum_{j=1}^{k-1}(t_j-t_k)\omega_j} e^{-\mathrm{i}t_k(\omega_1+\cdots+\omega_k)} \times \\
&\quad \times \mathrm{cum}\left(X_{t_1-t_k}(\tau_1),\ldots,X_{t_{k-1}-t_k},X_0(\tau_k)\right)
\end{aligned} \tag{B.2}$$

With the change of variables $t=t_k, u_j=t_j-t_k$ for $j=1,2,\ldots,k-1$, and defining $h^{(T)}(t)=1$ if $0\le t\le T$, and 0 otherwise, we can re-express the last expression as

$$\begin{aligned}
&(2\pi T)^{k/2}\mathrm{cum}\left(\widetilde{X}^{(T)}_{\omega_1}(\tau_1),\ldots,\widetilde{X}^{(T)}_{\omega_k}(\tau_k)\right)\\
&= \sum_{u_1,\ldots,u_{k-1}=-(T-1)}^{T-1} e^{-\mathrm{i}\sum_{j=1}^{k-1}u_j\omega_j}\,\mathrm{cum}\left(X_{u_1}(\tau_1),\ldots,X_{u_{k-1}},X_0(\tau_k)\right)\\
&\qquad \times \sum_{t\in\mathbb{Z}} h^{(T)}(u_1+t)\cdots h^{(T)}(u_{k-1}+t)h^{(T)}(t)e^{-\mathrm{i}t(\omega_1+\cdots+\omega_k)}\\
&= \sum_{u_1,\ldots,u_{k-1}=-(T-1)}^{T-1} e^{-\mathrm{i}\sum_{j=1}^{k-1}u_j\omega_j}\Delta^{(T)}\left(\sum_{j=1}^{k}\omega_j\right)\,\mathrm{cum}\left(X_{u_1}(\tau_1),\ldots,X_{u_{k-1}},X_0(\tau_k)\right)+\varepsilon_{1,T},
\end{aligned}$$

where $\Delta^{(T)}(\omega)=\sum_{t=0}^{T-1}e^{-\mathrm{i}\omega t}$. Now, $\varepsilon_{1,T}$ is an error term, that we can bound using Lemma F.7:

$$|\varepsilon_{1,T}| \le 2 \sum_{u_1,\ldots,u_{k-1}=-(T-1)}^{T-1} (|u_1|+\cdots+|u_{k-1}|)\left\|\mathrm{cum}\left(X_{u_1},\ldots,X_{u_{k-1}},X_0\right)\right\|_\infty.$$

Using the dominated convergence theorem, we find that $\varepsilon_{1,T}\sim o(T)$ under the first mixing condition, and $\varepsilon_{1,T}\sim O(1)$ under the second one, in both cases independently of ω. For the rest of the proof, we shall omit the τ_j's. Using Lemma B.1, we have

$$T^{k/2}\mathrm{cum}\left(\widetilde{X}_{\omega_1}^{(T)},\dots,\widetilde{X}_{\omega_k}^{(T)}\right) = (2\pi)^{k/2-1}\Delta^{(T)}\left(\sum_{j=1}^{k}\omega_j\right) f_{\omega_1,\dots,\omega_{k-1}} + (2\pi)^{k/2-1}\Delta^{(T)}\left(\sum_{j=1}^{k}\omega_j\right)\varepsilon_{2,T} + (2\pi)^{-k/2}\varepsilon_{1,T},$$

where $\varepsilon_{2,T}$ is the error term of Lemma B.1. Since $\Delta^{(T)}$ is $O(T)$, we obtain that the global error term is $o(T)$ (resp. $O(1)$) under the first (resp. second) cumulant mixing condition. □

APPENDIX C: THE PERIODOGRAM KERNEL AND ITS PROPERTIES

We recall the following condition used in the main paper:

Condition C(l,k): For each $j = 1,\dots,k-1$,

$$\sum_{t_1,\dots,t_{k-1}=-\infty}^{\infty}(1+|t_j|^l)\left\|\mathrm{cum}\left(X_{t_1},\dots,X_{t_{k-1}},X_0\right)\right\|_2 < \infty.$$

The cumulant mixing conditions of Theorem B.2 are simply **C(0,k)** and **C(1,k)**. With this definition in place, we may determine the exact mean of the periodogram kernel:

Proposition C.1. *Assuming that **C(0,2)** holds true, we have*

$$\mathbb{E}\left[p_{\omega}^{(T)}(\tau,\sigma)\right] = \int_{-\pi}^{\pi}F_T(\omega-\alpha)f_{\alpha}(\tau,\sigma)d\alpha + \mu(\tau)\mu(\sigma)F_T(\omega), \quad \textit{in } L^2.$$

In particular, if $\omega = 2\pi s/T$, with s an integer such that $s \not\equiv 0 \mod T$,

$$\mathbb{E}\left[p_{\omega}^{(T)}(\tau,\sigma)\right] = \int_{-\pi}^{\pi}F_T(\omega-\alpha)f_{\alpha}(\tau,\sigma)d\alpha, \quad \textit{in } L^2.$$

Proof.

$$\begin{aligned}\mathbb{E}p_{\omega}^{(T)}(\tau,\sigma) &= \frac{1}{2\pi}\mathrm{cov}\left(\widetilde{X}_{\omega}^{(T)},\widetilde{X}_{\omega}^{(T)}\right) + \frac{1}{2\pi}\mathbb{E}\left[\widetilde{X}_{\omega}^{(T)}(\tau)\right]\mathbb{E}\left[\widetilde{X}_{-\omega}^{(T)}(\sigma)\right]\\ &= \frac{1}{2\pi T}\sum_{s,t=0}^{T-1}e^{-\mathrm{i}\omega(t-s)}\mathrm{cov}(X_t(\tau),X_s(\sigma)) + \frac{1}{2\pi T}\mu(\tau)\mu(\sigma)|\Delta^{(T)}(\omega)|^2\\ &= \frac{1}{2\pi T}\sum_{s,t=0}^{T-1}e^{-\mathrm{i}\omega(t-s)}r_{t-s}(\tau,\sigma) + \mu(\tau)\mu(\sigma)F_T(\omega).\end{aligned}$$

Using the inversion formula of Lemma A.1, we obtain

$$
\begin{aligned}
&= \frac{1}{2\pi T}\sum_{s,t=0}^{T-1} e^{-\mathrm{i}\omega(t-s)}\int_{-\pi}^{\pi} e^{\mathrm{i}\alpha(t-s)} f_\alpha(\tau,\sigma)d\alpha + \mu(\tau)\mu(\sigma)F_T(\omega)\\
&= \frac{1}{2\pi T}\int_{-\pi}^{\pi}\underbrace{\left[\sum_{s,t=0}^{T-1} e^{-\mathrm{i}(t-s)(\omega-\alpha)}\right]}_{=|\Delta^{(T)}(\omega-\alpha)|^2} f_\alpha(\tau,\sigma)d\alpha + \mu(\tau)\mu(\sigma)F_T(\omega)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_T(\omega-\alpha) f_\alpha(\tau,\sigma)d\alpha + \mu(\tau)\mu(\sigma)F_T(\omega).
\end{aligned}
$$

For the case $\omega = 2\pi s/T$, with s an integer with $s \not\equiv 0 \mod T$, the result follows from the fact that $F_T(\omega) = 0$. $\square$

Theorem C.2. *Assume ω_1 and ω_2 are of the form $2\pi s(T)/T$, where $s(T)$ is an integer, $s(T) \not\equiv 0 \mod T$. We have*

$$
\begin{aligned}
\operatorname{cov}\left(p^{(T)}_{\omega_1}(\tau_1,\sigma_1), p^{(T)}_{\omega_2}(\tau_2,\sigma_2)\right) &= \eta(\omega_1-\omega_2) f_{\omega_1}(\tau_1,\tau_2) f_{-\omega_1}(\sigma_1,\sigma_2) +\\
&\quad + \eta(\omega_1+\omega_2) f_{\omega_1}(\tau_1,\sigma_2) f_{-\omega_1}(\sigma_1,\tau_2) + \varepsilon_T,
\end{aligned}
$$

*where the function $\eta(x)$ equals one if $x \in 2\pi\mathbb{Z}$, and zero otherwise. The error term ε_T is $o(1)$ under **C(0,2)** and **C(0,4)**; $\varepsilon_T \sim O(T^{-1})$ under **C(1,2)** and **C(1,4)**. In each case, the error term is uniform in ω_1, ω_2 of the form $2\pi s(T)/T$ with $s(T) \not\equiv 0 \mod T$.*

Proof. Since

$$
\begin{aligned}
&\operatorname{cov}\left(p^{(T)}_{\omega_1}(\tau_1,\sigma_1), p^{(T)}_{\omega_2}(\tau_2,\sigma_2)\right)\\
&= \operatorname{cov}\left(\widetilde{X}^{(T)}_{\omega_1}(\tau_1)\widetilde{X}^{(T)}_{-\omega_1}(\sigma_1), \widetilde{X}^{(T)}_{\omega_2}(\tau_2)\widetilde{X}^{(T)}_{-\omega_2}(\sigma_2)\right)\\
&= \mathbb{E}\left[\underbrace{\widetilde{X}^{(T)}_{\omega_1}(\tau_1)}_{A}\underbrace{\widetilde{X}^{(T)}_{-\omega_1}(\sigma_1)}_{B}\underbrace{\widetilde{X}^{(T)}_{-\omega_2}(\tau_2)}_{C}\underbrace{\widetilde{X}^{(T)}_{\omega_2}(\sigma_2)}_{D}\right]\\
&\quad - \mathbb{E}\left[\widetilde{X}^{(T)}_{\omega_1}(\tau_1)\widetilde{X}^{(T)}_{-\omega_1}(\sigma_1)\right]\mathbb{E}\left[\widetilde{X}^{(T)}_{-\omega_2}(\tau_2)\widetilde{X}^{(T)}_{\omega_2}(\sigma_2)\right]\\
&= \mathbb{E}[ABCD] - \mathbb{E}[AB]\mathbb{E}[CD],
\end{aligned}
$$

using the notation $(A) = \operatorname{cum}(A) = \mathbb{E}A$, $(A,B) = \operatorname{cum}(A,B)$, $(A,B,C) =$

cum (A,B,C), and so on, and invoking Lemma F.8, we deduce

$$\begin{aligned}
\operatorname{cov}\left(p^{(T)}_{\omega_1}(\tau_1,\sigma_1), p^{(T)}_{\omega_2}(\tau_2,\sigma_2)\right) &= (A,B,C,D)+\\
&+(A)(B,C,D)+(B)(A,C,D)+(C)(A,B,D)+(D)(A,B,C)\\
&+(A,C)(B,D)+(A,D)(B,C)\\
&+(A)(C)(B,D)+(A)(D)(B,C)+(B)(C)(A,D)+(B)(D)(A,C)
\end{aligned}$$

$$= (A,B,C,D)+(A,C)(B,D)+(A,D)(B,C). \tag{C.1}$$

The last equality comes from the fact that $(A)=(B)=(C)=(D)=0$, given that ω_1,ω_2 are of the form $2\pi s(T)/T$, with $s(T)\not\equiv 0 \mod T$. We now approximate each term of (C.1) using Theorem B.2:

$$(A,B,C,D)=\frac{2\pi}{T}f_{\omega_1,-\omega_1,-\omega_2}(\tau_1,\sigma_1,\tau_2,\sigma_2)+T^{-2}\varepsilon_T,$$

where ε_T is the error term of Theorem B.2. Thus $(A,B,C,D)=O(T^{-1})$, uniformly in ω under either **C(0,4)** or **C(1,4)**.

For the next term, we have under the assumption **C(1,2)/C(0,2)**:

$$\begin{aligned}
&(A,C)(B,D)=\\
&=T^{-2}\left\{\Delta^{(T)}(\omega_1-\omega_2)f_{\omega_1}(\tau_1,\tau_2)+\Big|^{o(T)}_{O(1)}\right\}\left\{\Delta^{(T)}(\omega_2-\omega_1)f_{-\omega_1}(\sigma_1,\sigma_2)+\Big|^{o(T)}_{O(1)}\right\}\\
&=T^{-2}|\Delta^{(T)}(\omega_1-\omega_2)|^2 f_{\omega_1}(\tau_1,\tau_2)f_{-\omega_1}(\sigma_1,\sigma_2)\\
&\quad+T^{-2}\Delta^{(T)}(\omega_1-\omega_2)f_{\omega_1}(\tau_1,\tau_2)\Big|^{o(T)}_{O(1)}\\
&\quad+T^{-2}\Delta^{(T)}(\omega_2-\omega_1)f_{-\omega_1}(\sigma_1,\sigma_2)\Big|^{o(T)}_{O(1)}+\Big|^{o(1)}_{O(T^{-2})}.
\end{aligned}$$

Using the fact that $\Delta^{(T)}=O(T), f_\omega=O(1)$ uniformly in ω, we obtain

$$(A,C)(B,D)=\eta(\omega_1-\omega_2)f_{\omega_1}(\tau_1,\tau_2)f_{-\omega_1}(\sigma_1,\sigma_2)+\Big|^{o(1)}_{O(T^{-1})},$$

where the function $\eta(x)$ equals one if $x\in 2\pi\mathbb{Z}$, and zero otherwise. A similar calculation yields

$$(A,D)(B,C)=\eta(\omega_1+\omega_2)f_{\omega_1}(\tau_1,\sigma_2)f_{-\omega_1}(\sigma_1,\tau_2)+\Big|^{o(1)}_{O(T^{-1})}$$

and in each of these cases, the error term is uniform in ω_1,ω_2 of the form $2\pi s(T)/T$, with $s(T)\not\equiv 0 \mod T$.

Piecing the results back together, we conclude that

$$\begin{aligned}\operatorname{cov}\left(p_{\omega_1}^{(T)}(\tau_1,\sigma_1), p_{\omega_2}^{(T)}(\tau_2,\sigma_2)\right) &= \eta(\omega_1-\omega_2)f_{\omega_1}(\tau_1,\tau_2)f_{-\omega_1}(\sigma_1,\sigma_2) \\ &\quad + \eta(\omega_1+\omega_2)f_{\omega_1}(\tau_1,\sigma_2)f_{-\omega_1}(\sigma_1,\tau_2) + \Big|_{O(T^{-1})}^{o(1)},\end{aligned}$$

with the error term being uniform in ω_1,ω_2 of the form $2\pi s(T)/T$, with $s(T) \not\equiv 0 \mod T$. □

APPENDIX D: ESTIMATION OF THE SPECTRAL DENSITY OPERATOR

Let $W(x)$ be a real function defined on $\mathbb{R}$ such that

1. W is positive, even, and bounded in variation,
2. $W(x) = 0$ if $|x| \geq 1$,
3. $\int_{-\infty}^{\infty} W(x)\mathrm{d}x = 1$,
4. $\int_{-\infty}^{\infty} W(x)^2\mathrm{d}x < \infty$.

The assumption of a compact support is not necessary, but will simplify proofs. For a bandwidth $B_T > 0$, write

$$W^{(T)}(x) = \sum_{j\in\mathbb{Z}} \frac{1}{B_T} W\left(\frac{x+2\pi j}{B_T}\right). \tag{D.1}$$

Some properties of W can be found in Section F. Our *spectral density estimator* $f_\omega^{(T)}$ of f_ω at frequency ω is the weighted average of the periodogram evaluated at frequencies of the form $\{2\pi s/T\}_{s=1}^{T-1}$, with weight function $W^{(T)}$:

$$f_\omega^{(T)}(\tau,\sigma) = \frac{2\pi}{T}\sum_{s=1}^{T-1} W^{(T)}\left(\omega - \frac{2\pi s}{T}\right) p_{\frac{2\pi s}{T}}^{(T)}(\tau,\sigma).$$

Concerning the mean of the spectral density estimator, we have:

Proposition D.1. *Under **C(1,2)**, if $B_T \to 0$ and $B_T T \to \infty$ as $T \to \infty$, then*

$$\mathbb{E}f_\omega^{(T)}(\tau,\sigma) = \int_{\mathbb{R}} W(x) f_{\omega - xB_T}(\tau,\sigma)\mathrm{d}x + O(T^{-1}) + O(B_T^{-1}T^{-1}), \quad \text{in } L^2,$$

and the error terms are uniform in ω.

PROOF. We use Proposition 2.6 to write

$$\mathbb{E}f_{\omega}^{(T)}(\tau,\sigma) = \frac{2\pi}{T}\sum_{s=1}^{T-1} W^{(T)}\left(\omega - \frac{2\pi s}{T}\right)\left\{f_{2\pi s}(\tau,\sigma) + O(T^{-1})\right\}$$

$$= \frac{2\pi}{T}\sum_{s=1}^{T-1} W^{(T)}\left(\omega - \frac{2\pi s}{T}\right) f_{2\pi s}(\tau,\sigma) + O(T^{-1})\left\{\frac{2\pi}{T}\sum_{s=1}^{T-1} W^{(T)}\left(\omega - \frac{2\pi s}{T}\right)\right\},$$

where the error term is uniform in s. Using Lemmas F.6, F.10 and F.12 we obtain

$$\frac{2\pi}{T}\sum_{s=1}^{T-1} W^{(T)}\left(\omega - \frac{2\pi s}{T}\right) f_{2\pi s}(\tau,\sigma) = \int_0^{2\pi} W^{(T)}(\omega-\alpha) f_\alpha(\tau,\sigma) d\alpha + \varepsilon_T,$$

where $\varepsilon_T \sim O(B_T^{-1}T^{-1})$, uniformly in ω. Using Lemma F.12,

$$\frac{2\pi}{T}\sum_{s=1}^{T-1} W^{(T)}\left(\omega - \frac{2\pi s}{T}\right) = O(1)$$

if $B_T T \to \infty$. Combining these facts, we may write

$$\mathbb{E}f_{\omega}^{(T)}(\tau,\sigma) = \int_0^{2\pi} W^{(T)}(\omega-\alpha) f_\alpha(\tau,\sigma) d\alpha + O(B_T^{-1}T^{-1}) + O(T^{-1}).$$

With a change of variables, we obtain

$$\int_0^{2\pi} W^{(T)}(\omega-\alpha) f_\alpha d\alpha = \int_{\mathbb{R}} W(x) f_{\omega - xB_T} dx.$$

This completes the proof. □

Concerning the covariance of the spectral density estimator, we have:

THEOREM D.2. *Under **C(1,2)** and **C(1,4)**,*

$$\begin{aligned} &\operatorname{cov}\left(f_{\omega_1}^{(T)}(\tau_1,\sigma_1), f_{\omega_2}^{(T)}(\tau_2,\sigma_2)\right) \\ &= \frac{2\pi}{T}\int_{-\pi}^{\pi}\Big\{W^{(T)}(\omega_1-\alpha)W^{(T)}(\omega_2-\alpha) f_\alpha(\tau_1,\tau_2) f_{-\alpha}(\sigma_1,\sigma_2) \\ &\quad + W^{(T)}(\omega_1-\alpha)W^{(T)}(\omega_2+\alpha) f_\alpha(\tau_1,\sigma_2) f_{-\alpha}(\sigma_1,\tau_2)\Big\} \mathrm{d}\alpha \\ &\quad + O(B_T^{-2}T^{-2}) + O(T^{-1}), \end{aligned}$$

in L^2, and the error terms are uniform in ω.

PROOF. Using Theorem C.2, conditions **C(1,2)** and **C(1,4)**, yield

$$
\begin{aligned}
&\operatorname{cov}\left(f_{\omega_1}^{(T)}(\tau_1,\sigma_1), f_{\omega_2}^{(T)}(\tau_2,\sigma_2)\right)\\
&= (2\pi/T)^2 \sum_{s,l=1}^{T-1} W^{(T)}(\omega_1 - 2\pi s/T) W^{(T)}(\omega_2 - 2\pi l/T)\times\\
&\quad \times \Big\{ \eta(2\pi(s-l)/T) f_{\frac{2\pi s}{T}}(\tau_1,\tau_2) f_{\frac{-2\pi s}{T}}(\sigma_1,\sigma_2) +\\
&\qquad\quad + \eta(2\pi(s+l)/T) f_{\frac{2\pi s}{T}}(\tau_1,\sigma_2) f_{\frac{-2\pi s}{T}}(\sigma_1,\tau_2) + O(T^{-1}) \Big\}\\
&= \left[\frac{2\pi}{T}\sum_{s=1}^{T-1} W^{(T)}(\omega_1 - 2\pi s/T)\right]^2 O(T^{-1})\\
&\quad + \left(\frac{2\pi}{T}\right)^2 \sum_{s=1}^{T-1} W^{(T)}(\omega_1 - 2\pi s/T) W^{(T)}(\omega_1 - 2\pi s/T) f_{\frac{2\pi s}{T}}(\tau_1,\tau_2) f_{\frac{-2\pi s}{T}}(\sigma_1,\sigma_2)\\
&\quad + \left(\frac{2\pi}{T}\right)^2 \sum_{s=1}^{T-1} W^{(T)}(\omega_1 - 2\pi s/T) W^{(T)}(\omega_1 + 2\pi s/T) f_{\frac{2\pi s}{T}}(\tau_1,\sigma_2) f_{\frac{-2\pi s}{T}}(\sigma_1,\tau_2),
\end{aligned}
$$

where the error term is uniform in s, l. An application of Lemmas F.6, F.10, F.11 and F.12 now completes the proof. ☐

APPENDIX E: A CLASS OF EXAMPLES

Our central result concerning the asymptotic representation of the discrete Fourier transform requires mixing conditions on the underlying stationary process that might be non-trivial to verify in certain situations. However, in this section, we demonstrate that these conditions are valid for a broad class of stationary processes, namely linear processes of any order (potentially of infinite order), under natural conditions on their coefficient operators and innovation processes. For tidiness, we will employ the same notation for norms on functions $f : [0,1] \to \mathbb{C}$ or $g : [0,1]\times[0,1] \to \mathbb{C}$, as it will usually be clear which is the case by the context:

$$
\|f\|_p = \left(\int_0^1 |f(\sigma)|^p d\sigma\right)^{1/p}, \quad \|g\|_p = \left(\iint_{[0,1]^2} |g(\tau,\sigma)|^p d\tau d\sigma\right)^{1/p},
$$

$$
\|f\|_\infty = \sup_{\tau\in[0,1]} |f(\tau)|, \qquad \|g\|_\infty = \sup_{\tau,\sigma\in[0,1]} |g(\tau,\sigma)|.
$$

We will also use the following notation for the norm with respect to one of the variables: $\|g(\tau,\cdot)\|_p = \left(\int_0^1 |g(\tau,\sigma)|^p d\sigma\right)^{1/p}$, and similarly for $\|g(\tau,\cdot)\|_\infty$.

With these definitions in place, we can provide the following moment conditions on a general linear process, which are sufficient for the mixing conditions to hold:

PROPOSITION E.1. *Let*

$$X_t = \sum_{s \in \mathbb{Z}} A_s \varepsilon_{t-s}$$

be a linear process with $\mathbb{E}\|\varepsilon_0\|_2^p < \infty$ *for all* $p \geq 1$, *and*

$$\sum_{s \in \mathbb{Z}} (1 + |s|^l) \|a_s\|_2 < \infty$$

for some positive integer l, *where* a_s *is the kernel of* A_s. *Then for all fixed* $k = 1, 2, \ldots$, X_t *satisfies* ***C(l,k)****:*

$$\sum_{t_1,\ldots,t_{k-1} \in \mathbb{Z}} (1 + |t_j|^l) \|\mathrm{cum}\,(X_{t_1}, \ldots, X_{t_{k-1}}, X_0)\|_2 < \infty, \quad j = 1, \ldots, k-1. \tag{E.1}$$

In addition,

$$\sum_{t \in \mathbb{Z}} (1 + |t|^l) |||\mathscr{R}_t|||_1 < \infty, \tag{E.2}$$

and,

$$\sum_{t_1,t_2,t_3} |||\mathrm{cum}\,(X_{t_1}, X_{t_2}, X_{t_3}, X_0)|||_1 < \infty, \tag{E.3}$$

where we view $\mathrm{cum}\,(X_{t_1}, X_{t_2}, X_{t_3}, X_0)$ *as an operator on* $L^2([0,1]^2, \mathbb{R})$ *– see Section 2.*

PROOF OF PROPOSITION E.1. We first prove (E.1). Since the assumptions of Proposition F.14 are satisfied, we have

$$\mathrm{cum}\,(X_{t_1}, \ldots, X_{t_{k-1}}, X_0) = \sum_{s \in \mathbb{Z}} (A_{t_1 - s} \otimes \cdots \otimes A_{t_{k-1}-s} \otimes A_{-s}) \mathrm{cum}\,(\varepsilon_0)_k\,,$$

where the convergence is in L^2, and

$$\mathrm{cum}\,(\varepsilon_0)_k = \mathrm{cum}\left(\underbrace{\varepsilon_0, \ldots, \varepsilon_0}_{k \text{ times}}\right).$$

Since

$$\begin{aligned}&\big\|(A_{t_1-s}\otimes\cdots\otimes A_{t_{k-1}-s}\otimes A_{-s})\mathrm{cum}\,(\varepsilon_0)_k\big\|_2\\&\qquad\le\|a_{t_1-s}\|_2\cdots\big\|a_{t_{k-1}-s}\big\|_2\|a_{-s}\|_2\|\mathrm{cum}\,(\varepsilon_0)_k\|_2,\end{aligned}$$

we have

$$\sum_{t_1,\ldots,t_{k-1}\in\mathbb{Z}}\big\|\mathrm{cum}\,(X_{t_1},\ldots,X_{t_{k-1}},X_0)\big\|_2\le\left(\sum_{t\in\mathbb{Z}}(1+|t|^l)\|a_t\|_2\right)^k\|\mathrm{cum}\,(\varepsilon_0)_k\|_2.$$

Furthermore,

$$\begin{aligned}&\sum_{t_1,\ldots,t_{k-1}\in\mathbb{Z}}|t_j|^l\big\|\mathrm{cum}\,(X_{t_1},\ldots,X_{t_{k-1}},X_0)\big\|_2\\&\le\sum_{t_1,\ldots,t_{k-1},s\in\mathbb{Z}}|t_j|^l\|a_{t_1-s}\|_2\cdots\big\|a_{t_{k-1}-s}\big\|_2\|a_{-s}\|_2\|\mathrm{cum}\,(\varepsilon_0)_k\|_2\end{aligned}$$

and, using the change of variables $u_1=t_1-s,\ldots,u_{k-1}=t_{k-1}-s$, $u_k=-s$, and Jensen's inequality, we obtain

$$\le 2^l\left(\sum_{u\in\mathbb{Z}}(1+|u|)^l\|a_u\|_2\right)^k\|\mathrm{cum}\,(\varepsilon_0)_k\|_2.$$

Therefore

$$\sum_{t_1,\ldots,t_{k-1}\in\mathbb{Z}}(1+|t_j|^l)\big\|\mathrm{cum}\,(X_{t_1},\ldots,X_{t_{k-1}},X_0)\big\|_2<\infty.$$

We now turn to the proof of (E.2). By Proposition F.14,

$$\mathscr{R}_t=\sum_{s\in\mathbb{Z}}(A_{t-s}\otimes A_{-s})\mathrm{cum}\,(\varepsilon_0)_2\,.$$

Using Lemma F.17, we have

$$(A_{t-s}\otimes A_{-s})\mathrm{cum}\,(\varepsilon_0)_2=A_{t-s}\mathrm{cum}\,(\varepsilon_0)_2\,A^*_{-s},$$

and hence Proposition F.20 yields

$$|||\mathscr{R}_t|||_1\le|||\mathrm{cum}\,(\varepsilon_0)_2|||_1\sum_{s\in\mathbb{Z}}|||A_{t-s}|||_\infty|||A_{-s}|||_\infty.$$

Since $\mathrm{cum}\,(\varepsilon_0)_2$ is the covariance operator of the random element ε_0, Lemma F.22 implies that

$$|||\mathrm{cum}\,(\varepsilon_0)_2|||_1=\mathbb{E}\|\varepsilon_0-\mathbb{E}\varepsilon_0\|_2^2<\infty.$$

Furthermore, since A_t is an integral operator,

$$|||A_t|||_\infty \leq \|a_t\|_2.$$

Hence

$$\sum_{t\in\mathbb{Z}} |||\mathscr{R}_t|||_1 \leq \mathbb{E}\|\varepsilon_0 - \mathbb{E}\varepsilon_0\|_2^2 \sum_{t,s\in\mathbb{Z}} \|a_{t-s}\|_2 \|a_{-s}\|_2 < \infty$$

Using the same arguments as before, we also obtain

$$\sum_t (1+|t|^l)|||\mathscr{R}_t|||_1 < \infty.$$

We now turn to show (E.3). Recall that

$$\mathrm{cum}\,(X_{t_1}, X_{t_2}, X_{t_3}, X_{t_4}) = \sum_{s\in\mathbb{Z}} (A_{t_1-s} \otimes \cdots \otimes A_{t_4-s})\mathrm{cum}\,(\varepsilon_0)_4\,.$$

Notice that since $(A_t \otimes A_s)$ is an operator on $L^2([0,1]^2, \mathbb{R})$,

$$((A_{t_1-s} \otimes A_{t_2-s}) \bigotimes (A_{t_3-s} \otimes A_{t_4-s}))$$

is an operator on $L^2([0,1]^4, \mathbb{R})$ (see Lemma F.17) which is in fact equal to

$$(A_{t_1-s} \otimes \cdots \otimes A_{t_4-s}).$$

Hence if we view $\mathrm{cum}\,(\varepsilon_0)_4$ as an integral operator on $L^2([0,1]^2, \mathbb{R})$, with kernel

$$k((\tau_1,\tau_2),(\sigma_1,\sigma_2)) = \mathrm{cum}\,(\varepsilon_0(\tau_1), \varepsilon_0(\tau_2), \varepsilon_0(\sigma_1), \varepsilon_0(\sigma_2))\,,$$

we obtain

$$(A_{t_1-s} \otimes \cdots \otimes A_{t_4-s})\mathrm{cum}\,(\varepsilon_0)_4 = (A_{t_1-s} \otimes A_{t_2-s})\mathrm{cum}\,(\varepsilon_0)_4\,(A^*_{t_3-s} \otimes A^*_{t_4-s}).$$

Thus, Proposition F.20 yields

$$|||(A_{t_1-s} \otimes \cdots \otimes A_{t_4-s})\mathrm{cum}\,(\varepsilon_0)_4|||_1 \leq |||A_{t_1-s} \otimes A_{t_2-s}|||_2 |||\mathrm{cum}\,(\varepsilon_0)_4|||_2 |||A^*_{t_3-s} \otimes A^*_{t_4-s}|||_2.$$

Since $\mathbb{E}\|\varepsilon_0\|_2^4 < \infty$, Proposition F.14 implies

$$|||\mathrm{cum}\,(\varepsilon_0)_4|||_2 = \|\mathrm{cum}\,(\varepsilon_0)_4\|_2 < \infty.$$

We also have

$$|||A_{t_1} \otimes A_{t_2}|||_2 = \|a_{t_1}\|_2 \|a_{t_2}\|_2,$$

and

$$|||A^*_{t_3} \otimes A^*_{t_4}|||_2 = \|a_{t_3}\|_2 \|a_{t_4}\|_2.$$

Hence

$$\sum_{t_1,t_2,t_3} |||\mathrm{cum}\,(X_{t_1}, X_{t_2}, X_{t_3}, X_0)|||_1 < \infty.$$

□

APPENDIX F: BACKGROUND RESULTS AND TECHNICAL STATEMENTS

This section contains several intermediate results in functional analysis and probability in function space that are required in our earlier formal derivations. In addition, we also collect some known results and facts for the reader's ease. Our first two lemmas provide an easily verifiable L^2 moment condition that is sufficient for tightness to hold true. They collect arguments appearing in the proof of Bosq (2000, Theorem 2.7):

LEMMA F.1 (A class of compact sets for separable Hilbert spaces). *Let H be a separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_2$. For any sequence of integers $1 = n_1 < n_2 < \cdots$, any sequence of positive numbers l_k such that $\lim_{k\to\infty} l_k = +\infty$, and any complete orthonormal sequence $(e_n)_{n=1,2,\ldots}$ of H, the set $K = \bigcap_{k=1}^{\infty} B_k$ is compact, where $B_k = \left\{x \in H : \sum_{j=n_k}^{\infty} \langle x, e_j\rangle^2 \leq l_k^{-1}\right\}$.*

PROOF. Suppose H is a real separable Hilbert space (all the following steps can be reproduced for a complex separable Hilbert space). The sequence (e_n) induces an isometric isomorphism $H \to \ell_2(\mathbb{R})$ via

$$x \in H \longmapsto (\langle x, e_1\rangle, \langle x, e_2\rangle, \ldots).$$

We can therefore write $x = (x_1, x_2, \ldots)$, where $x_j = \langle x, e_j\rangle$.

Since K is in particular a metric space, showing its compactness is equivalent to showing that it is complete and totally bounded (Munkres 2000, Theorem 45.1). Recall that a metric space (M, d) is totally bounded if, for every $\varepsilon > 0$, there exists a *finite* covering of M by ε-balls. This means that there exists a subset $F_\varepsilon \subset M$ such that

$$\text{for any point of } m \in M, \text{ there is a point } p \in F_\varepsilon \text{ with } d(m,p) < \varepsilon. \tag{F.1}$$

The set F_ε is called a (finite) ε-net of M.

First notice that K is complete since it is a closed subset of H. Let us show that is is totally bounded. Fix $\varepsilon > 0$, and let k be the smallest integer such that $l_k^{-1} < \varepsilon^2/2$. Let

$$H_k = \{(x_1, x_2, \ldots) \in H : x_j = 0 \text{ for } j > k\}.$$

Since the set

$$H_k(l_1^{-1/2}) = \left\{x \in H_k : \|x\|_2 < l_1^{-1/2}\right\}$$

is compact, it is totally bounded, and there exists a finite $\frac{\varepsilon}{\sqrt{2}}$-net F of $H_k(l_1^{-1/2})$. Let us show that F is a (finite) ε-net of K. Let $x = (x_1, x_2, \dots) \in K$, $P_k : H \to H$ denote the orthogonal projection onto H_k, and $I : H \to I$ be the identity operator on H. Notice that

$$P_k(K) \subset P_k(B_1) = H_k(l_1^{-1/2}),$$

hence there exists a point $p \in F$ such that $\|P_k x - p\|_2^2 < \frac{\varepsilon^2}{2}$. Notice also that $\|(I - P_k)x\|_2^2 < l_k^{-1} < \frac{\varepsilon^2}{2}$ by definition of B_k. Hence

$$\|x - p\|_2^2 = \|P_k(x-p)\|_2^2 + \|(I - P_k)(x-p)\|_2^2 = \|P_k x - p\|_2^2 + \|(I-P_k)x\|_2^2,$$

since $p \in H_k$. We thus have $\|x - p\|_2 < \varepsilon$, and K is compact. □

LEMMA F.2 (Criterion for tightness in Hilbert Space). *Let H be a (real or complex) separable Hilbert Space, and $X_T : \omega \to H$, $T = 1, 2, \dots$ be a sequence of random variables. If for some fixed basis (e_n) of H, we have $\mathbb{E}\,|\langle X_T, e_n\rangle|^2 \le a_n, n = 1, 2, \dots;$ for all large T, and $\sum_{n\ge 1} a_n < \infty$, then (X_T) is tight.*

PROOF. Fix $\varepsilon > 0$. We shall define a compact set $K \subset H$ such that $\mathbb{P}(X_T \notin K) \le \varepsilon$, for all large T. This will show that X_T is tight.

Set $S_n^T = \sum_{j \ge n} \mathbb{E}|\langle X_T, e_j\rangle|^2$. For T large enough, $S_n^T \le S_n$, where

$$S_n = \sum_{j \ge n} a_j < \infty.$$

Notice that $\lim_{n\to\infty} S_n = 0$. Set $n_1 = 1$, and $l_1 = \varepsilon/(2S_1)$. We then define $l_k = k l_1$ and choose integers $1 < n_2 < n_3 < \cdots$ such that

$$S_{n_k} \le \frac{S_1}{k 2^{k-1}}.$$

Define $B_k = \left\{ x \in H : \sum_{j=n_k}^{\infty} \langle x, \varphi_j\rangle^2 \le l_k^{-1} \right\}$, and $K = \bigcap_{k=1}^{\infty} B_k$, which is compact by Lemma F.1. Using successively the union bound and Markov's inequality, we obtain

$$\begin{aligned} \mathbb{P}\left[X_T \notin K\right] &\le \sum_{k \ge 1} \mathbb{P}\left[\sum_{j \ge n_k} |\langle X_T, e_j\rangle|^2 \ge l_k^{-1} \right] \\ &\le \sum_k l_k \sum_{j \ge n_k} \mathbb{E}|\langle X_T, e_j\rangle|^2 \\ &\le \sum_k l_k S_{n_k} \le \sum_k \frac{\varepsilon}{2^k} = \varepsilon. \end{aligned}$$

□

LEMMA F.3. *Let* $p_\omega^{(T)}(\tau,\sigma) = T^{-1}\sum_{t,s=0}^{T-1} e^{-\mathrm{i}\omega(t-s)} X_t(\tau)X_s(\sigma) = \widetilde{X}_\omega^{(T)}(\tau)\widetilde{X}_{-\omega}^{(T)}(\sigma)$. *Then,* $\mathbb{E}p_\omega^{(T)} = T^{-1}(a_0 + \cdots + a_{T-1})$, *where* $a_T(\tau,\sigma) = \sum_{t=-T}^{T} e^{-\mathrm{i}\omega t} r_t(\tau,\sigma)$.

PROOF. Make the change of variables $u = t - s$, and observe that, by stationarity of X_t,

$$\mathbb{E}p_\omega^{(T)}(\tau,\sigma) = T^{-1}\sum_{u=-(T-1)}^{T-1}(T-|u|)e^{-\mathrm{i}\omega u} r_u(\tau,\sigma),$$

which then yields the result. □

LEMMA F.4. *If* $\int_{-\infty}^{\infty}|x|^p W(x)dx < \infty$ *and* ***C(p,2)*** *holds true, then*

$$\int_{\mathbb{R}} W(x) f_{\omega - xB_T} dx = f_\omega + \\ + \sum_{k=1}^{p-1}\frac{(-1)^k B_T^k}{k!}\frac{\partial^k f_\omega}{\partial\omega^k}\cdot\int_{\mathbb{R}} x^k W(x)dx + O(B_T^p),$$

in L^2, *and the error term is uniform in* ω. *Notice that since* W *is even, the integral is zero if* k *is odd. The case* $p = 1$ *will be useful for consistent estimation of* f_ω:

$$\int W(x) f_{\omega - xB_T} dx = f_\omega + O(B_T), \quad \text{in } L^2,$$

the error term being uniform in ω.

PROOF. In the following, although all equalities are meant in the L^2 sense with respect to the variables τ, σ. Since for every $\varphi \in L^2([0,1],\mathbb{C})$, the mapping

$$\omega \mapsto \frac{\partial^k}{\partial\omega^k}\langle f_\omega, \varphi\rangle$$

is continuous, we can write the Taylor expansion of $f_{\omega - xB_T}(\tau,\sigma)$, with respect to x at $x = 0$:

$$f_{\omega - xB_T}(\tau,\sigma) = f_\omega(\tau,\sigma) - B_T \frac{\partial f_\alpha(\tau,\sigma)}{\partial\alpha}\Big|_{\alpha=\omega} x + \cdots \\ + \frac{(-1)^{p-1}B_T^{p-1}}{(p-1)!}\frac{\partial^{p-1} f_\alpha(\tau,\sigma)}{\partial\alpha^{p-1}}\Big|_{\alpha=\omega} x^{p-1} \\ + R_p(x,\omega,\tau,\sigma),$$

where

$$R_p(x,\omega,\tau,\sigma) = \frac{(-1)^p B_T^p}{p!} \frac{\partial^p f_\alpha(\tau,\sigma)}{\partial \alpha^p}\Big|_{\alpha=\omega-\theta_x B_T} x^p,$$

and $\theta_x \in [0,x]$. This expression is bounded in L^2 by

$$\|R_p(x,\omega,\cdot,\cdot)\|_2 \leq \frac{B_T^p}{p!} \sup_\omega \left\| \frac{\partial^p}{\partial \omega^p} f_\omega \right\|_2 |x|^p,$$

which does not depend on ω. Hence we obtain

$$\int_{\mathbb{R}} W(x) f_{\omega - xB_T} dx = f_\omega + \sum_{k=1}^{p-1} \frac{(-1)^k B_T^k}{k!} \frac{\partial^k f_\omega}{\partial \omega^k} \cdot \int_{\mathbb{R}} x^k W(x) dx + \underbrace{\frac{B_T^p}{p!} \sup_\omega \left\| \frac{\partial^p}{\partial \omega^p} f_\omega \right\|_2 \int_{\mathbb{R}} |x|^p W(x) dx}_{=O(B_T^p)},$$

and the error is uniform in ω. □

Lemma F.5. *Let $x_1,\ldots,x_n; y_1,\ldots,y_n$ be (complex or real) numbers bounded by K. Then $|x_1 x_2 \cdots x_n - y_1 y_2 \cdots y_n| \leq K^{n-1} \sum_{j=1}^n |x_j - y_j|$.*

Proof. Rewriting the expression in a suitable way yields the result:

$$\begin{aligned} |x_1 x_2 \cdots x_n - y_1 y_2 \cdots y_n| &= \left| \sum_{k=1}^n \left(\prod_{j=1}^{k-1} y_j \prod_{l=k}^n x_l - \prod_{j=1}^k y_j \prod_{l=k+1}^n x_l \right) \right| \\ &\leq \sum_{k=1}^n \prod_{j=1}^{k-1} \prod_{l=k+1}^n |y_j x_l| |x_k - y_k| \\ &\leq K^{n-1} \sum_{k=1}^n |x_k - y_k|. \end{aligned}$$

□

Denote by $V_a^b(h)$ the total variation of a function $h : [a,b] \to \mathbb{C}$ (Wheeden & Zygmund 1977, Chapter 2.1).

Lemma F.6. *Let $f, f_1, \ldots, f_n : [a,b] \to \mathbb{C}$ be bounded in variation and bounded. Let $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$. Then,*

(i) $V_a^b \left(\prod_{j=1}^n f_j \right) \leq \sum_{i=1}^n V_a^b(f_i) \prod_{j\neq i} \|f_j\|_\infty$.

(ii) *If* $\psi : [c,d] \to [a,b]$ *is a strictly increasing bijection,* $V_a^b(f) = V_{\psi(c)}^{\psi(d)}(f) = V_c^d(f \circ \psi)$*. If* $\psi : [c,d] \to [a,b]$ *is a strictly decreasing bijection,* $V_a^b(f) = V_d^c(f \circ \psi)$*.*

(iii) *For any* $a < c < b$*, we have* $V_a^c(f) + V_c^b(f) = V_a^b(f)$*, and hence for any* $a \le c \le d \le b$*, we have* $V_c^d(f) \le V_a^b(f)$*.*

(iv) *For any* $\lambda \in \mathbb{C}$*,* $V_a^b(\lambda f) = |\lambda| V_a^b(f)$*.*

(v) *(triangle inequality)* $V_a^b(f_1 + f_2) \le V_a^b(f_1) + V_a^b(f_2)$*.*

(vi) *If* f *is continuous on* $[a,b]$*,* f' *exists on* (a,b) *and is Riemann integrable on* $[a,b]$*,* $V_a^b(f) = \int_a^b |f'(x)| dx$*.*

(vii) *If* $f : \mathbb{R} \to \mathbb{C}$ *is* 2π*-periodic, and* $g(x) = f(\omega - x)$ *for some* $\omega \in \mathbb{R}$*,* $V_0^{2\pi}(g) = V_0^{2\pi}(f)$*.*

(viii) *If* $V_a^b(f) < \infty$*,* f *is bounded on* $[a,b]$*.*

We notice that the total variation has some of the properties of a norm.

Proof. Recall the definition of total variation: let $\Gamma = \{x_0, \ldots, x_m\}$ be a partition of $[a,b]$, that is,

$$a = x_0 < x_1 < \cdots < x_m = b,$$

and define

$$S_\Gamma(f) = \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| .$$

The total variation of f between a and b is

$$V(f) = \sup_{\Gamma} \; S_\Gamma(f),$$

where the supremum is taken over all partitions Γ of $[a,b]$.

Hence for any such partitions, and any $f, g : [a,b] \to \mathbb{C}$,

$$\begin{aligned}
&\sum_{i=1}^{m} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&\le \sum_{i=1}^{m} |f(x_i)\left[g(x_i) - g(x_{i-1})\right]| \cdot |g(x_{i-1})\left[f(x_i) - f(x_{i-1})\right]| \\
&\le \|f\|_\infty \sum_{i=1}^{m} |g(x_i) - g(x_{i-1})| + \|g\|_\infty \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| \\
&\le \|f\|_\infty V(g) + \|g\| V(f).
\end{aligned}$$

Taking the supremum over all partitions yields

$$V(f \cdot g) \le \|f\|_\infty V(g) + \|g\|_\infty V(f).$$

We then obtain (i) by induction.

For (ii), first assume that ψ is a strictly increasing bijection. Therefore $\{x_0, \ldots, x_m\}$ is a partition of $[c,d]$ if and only if $\{\psi(x_0), \ldots, \psi(x_m)\}$ is a partition of $[a,b]$. If ψ is a strictly decreasing bijection, $\{x_0, \ldots, x_m\}$ is a partition of $[c,d]$ if and only if $\{\psi(x_m), \psi(x_{m-1}), \ldots, \psi(x_0)\}$ is a partition of $[a,b]$. Statement (ii) is a direct consequence of these one-to-one correspondences between the partitions of $[a,b]$ and $[c,d]$.

Statements $(iii), (vi)$ and $(viii)$ are proven in Wheeden & Zygmund (1977, Chapter 2.1).

For (iv), notice that

$$\sum_{i=1}^{m} |(\lambda f)(x_i) - (\lambda f)(x_{i-1})| = |\lambda| \cdot \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| \, .$$

For (v), notice that

$$|(f_1 + f_2)(x_i) - (f_1 + f_2)(x_{i-1})| \leq |f_1(x_i) - f_1(x_{i-1})| + |f_2(x_i) - f_2(x_{i-1})| \, .$$

Taking the sum over i and the supremum on all partitions $\{x_0, \ldots, x_m\}$ of $[a,b]$ yields the (v).

For (vii), define $g(x) = f(\omega - x)$. Notice that $V_a^b(f) = V_{a+2\pi}^{b+2\pi}(f)$ because f is 2π-periodic. Hence without loss of generality, we can assume that $0 < \omega < 2\pi$. Using (iii) we obtain

$$V_0^{2\pi}(f) = V_0^{\omega}(f) + V_{\omega}^{2\pi}(f) = V_{2\pi}^{\omega+2\pi}(f) + V_{\omega}^{2\pi}(f) = V_{\omega}^{\omega+2\pi}(f).$$

Choosing $\psi(x) = \omega - x$, (ii) yields

$$V_{\omega}^{\omega+2\pi}(f) = V_{-2\pi}^{0}(g) = V_0^{2\pi}(g).$$

□

We introduce the following condition, which will be used in the following results:

T1 (Taper condition 1) Let $h(u), -\infty < u < \infty$ be a real function, which is bounded, bounded in variation and with $h(u) = 0$ for $|u| \geq 1$. We denote by $\|h\|_\infty$ its supremum, and by $V_{-1}^{1}(h)$ its total variation (between -1 and 1).

Lemma F.7. *Suppose $h_{a_1}, \ldots, h_{a_k}$ satisfy **T1**, set $h_{a_j}^{(T)}(t) = h_{a_j}(t/T)$ and define $H_{a_1,\ldots,a_k}^{(T)}(\omega) = \sum_{t\in\mathbb{Z}} \left[\prod_{j=1}^{k} h_{a_j}^{(T)}(t)\right] \exp(-\mathbb{i}\omega t)$. We have the following* *inequality for all* $u_1, \dots, u_{k-1} \in \mathbb{Z}$*:*

$$\left|\sum_{t\in\mathbb{Z}} h_{a_1}^{(T)}(t+u_1)\cdots h_{a_{k-1}}^{(T)}(t+u_{k-1})h_{a_k}^{(T)}(t)\exp(-\mathrm{i}\omega t) - H_{a_1,\dots,a_k}^{(T)}(\omega)\right| \\ \leq K\Big(|u_1| + \cdots + |u_{k-1}|\Big),$$

where $K = \Big(\max_{j=1,\dots,k} \|h_{a_j}\|_\infty\Big)^{k-1} \cdot \Big(\max_{j=1,\dots,k} V_{-1}^{1}(h_{a_j})\Big)$ *is independent of* ω*.*

Proof. Using Lemma F.5, we obtain

$$\begin{aligned}&\left|\sum_{t\in\mathbb{Z}} h_{a_1}^{(T)}(t+u_1)\cdots h_{a_{k-1}}^{(T)}(t+u_{k-1})h_{a_k}^{(T)}(t)\exp(-\mathrm{i}\omega t) - H_{a_1,\dots,a_k}^{(T)}(\omega)\right| \\ &\leq \sum_{t\in\mathbb{Z}} \left|h_{a_k}^{(T)}(t)\right| \cdot \left|h_{a_1}^{(T)}(t+u_1)\cdots h_{a_{k-1}}^{(T)}(t+u_{k-1}) - h_{a_1}^{(T)}(t)\cdots h_{a_{k-1}}^{(T)}(t)\right| \\ &\leq \|h_{a_1}\|_\infty \left(\max_{j=1,\dots,k-1} \|h_{a_j}\|_\infty\right)^{k-2} \cdot \sum_{j=1}^{k-1}\sum_{t\in\mathbb{Z}} \left|h_{a_j}^{(T)}(t+u_j) - h_{a_1}^{(T)}(t)\right|.\end{aligned}$$

Let us now bound $\sum_{t\in\mathbb{Z}} \left|h_{a_j}^{(T)}(t+u_j) - h_{a_1}^{(T)}(t)\right|$. For simplicity, we suppress the indices. If $u > 0$,

$$\begin{aligned}\sum_{t\in\mathbb{Z}} \left|h_a^{(T)}(t+u) - h_a^{(T)}(t)\right| &\leq \sum_{v=0}^{u_j-1}\sum_{t\in\mathbb{Z}} \left|h_a^{(T)}(t+v+1) - h_a^{(T)}(t+v)\right| \\ &\leq \sum_{v=0}^{u_j-1} V_{-T}^{T}(h_a^{(T)}) \\ &= |u_j| V_{-1}^{1}(h_a),\end{aligned}$$

where we have used Lemma F.6 for the last equality. If $u < 0$, we simply replace $\sum_{v=0}^{u_j-1}$ with $\sum_{u_j+1}^{0}$, and the same bound holds. Hence

$$\left|\sum_{t\in\mathbb{Z}} h_{a_1}^{(T)}(t+u_1)\cdots h_{a_{k-1}}^{(T)}(t+u_{k-1})h_{a_k}^{(T)}(t)\exp(-\mathrm{i}\omega t) - H_{a_1,\dots,a_k}^{(T)}(\omega)\right| \\ \leq K\Big(|u_1| + \cdots + |u_{k-1}|\Big).$$

□

We recall the definition of a cumulant:

$$\mathrm{cum}\,(Y_1,\dots,Y_r) = \sum_{\nu}(-1)^{p-1}(p-1)!\,\prod_{l=1}^{p}\mathbb{E}\left[\prod_{j\in\nu_l}Y_j\right],$$

where the summation extends over all unordered partitions $\nu=(\nu_1,\dots,\nu_p), p=1,\dots,r$, of $\{1,\dots,r\}$. The following result is found in Rosenblatt (1985, p.34):

Lemma F.8. *If $\mathbb{E}|\prod_{j\in J}Y_j|<\infty$ for all subset of indices $J\subset\{1,\dots,r\}$,*

$$\mathbb{E}\,[Y_1\cdots Y_r] = \sum_{\nu}\prod_{l=1}^{p}\mathrm{cum}\,(Y_j; j\in\nu_l)\,,$$

where the sum extends over all unordered partitions $\nu=(\nu_1,\dots,\nu_p)$ of $\{1,\dots,r\}$.

Let $\langle\cdot,\cdot\rangle$ denote the scalar product in $L^2([0,1]^k)$, and $\|\cdot\|_2$ be the induced norm. We abuse notation in the following Lemma and do not distinguish the notation of the scalar product and norm for distinct k's.

Lemma F.9. *Let X_t be a strictly stationary functional time series, $X_t\in L^2([0,1],\mathbb{R})$ such that $\mathbb{E}\|X_0\|_2^p<\infty$ for all positive integers p. Then, for any $\psi_1,\dots,\psi_k\in L^2([0,1],\mathbb{R})$,*

$$\langle\mathrm{cum}\,(X_{t_1},\dots,X_{t_k})\,,\psi_1\otimes\cdots\otimes\psi_k\rangle = \mathrm{cum}\,(\langle X_{t_1},\psi_1\rangle,\dots,\langle X_{t_k},\psi_k\rangle)\,.$$

Hence

$$|\mathrm{cum}\,(\langle X_{t_1},\psi_1\rangle,\dots,\langle X_{t_k},\psi_k\rangle)\,| \le \|\mathrm{cum}\,(X_{t_1},\dots,X_{t_k})\|_2\|\psi_1\|_2\cdots\|\psi_k\|_2.$$

Proof. The proof is merely an application of Tonelli's theorem — which is justified since all strong moments of X_0 exist — and is therefore omitted. □

We shall require the following lemma to quantify the approximation error of integrals by Riemann sums.

Lemma F.10. *Let $h:[a,b]\to\mathbb{R}$ be a function of bounded variation. If $\Delta_n=\int_a^b h(t)dt-\frac{b-a}{n}\sum_{j=1}^n h(a+(b-a)j/n)$, then $|\Delta_n|\le(b-a)\frac{V_a^b(h)}{n}$. If the sum goes only from 1 to $(n-1)$, $|\Delta_n|\le(b-a)\frac{V_a^b(h)+h(b)}{n}$.*

PROOF. First let us prove the Lemma for $f : [0,1] \to \mathbb{R}$ of bounded variation. Since f is of bounded variation, it is bounded and has a finite number of discontinuities, and is hence Riemann integrable on $[0,1]$ (Wheeden & Zygmund 1977, see results (2.1), (2.8) and (5.54)). The following comes from Pólya & Szegö (1972, Pt.2, Chapter 1, problem 9). Set $\Delta'_n = \int_0^1 f(x)dx - \frac{1}{n}\sum_{j=1}^n f(j/n)$. Since

$$\int_0^1 f(x)dx = \sum_{j=1}^n \int_{(j-1)/n}^{j/n} f(x)dx = \sum_{j=1}^n \int_0^{1/n} f\left(\frac{j-1}{n} + x\right) dx,$$

we obtain

$$\begin{aligned}|\Delta'_n| &= \int_0^{1/n} \sum_{j=1}^n \left| f(j/n) - f\left(\frac{j-1}{n} + x\right)\right| dx \\ &\leq \int_0^{1/n} \sum_{j=1}^n \left\{ \left| f(j/n) - f\left(\frac{j-1}{n} + x\right)\right| + \left| f\left(\frac{j-1}{n} + x\right) - f\left(\frac{j-1}{n}\right)\right| \right\} dx \\ &\leq \int_0^{1/n} V_0^1(f)dx = \frac{V_0^1(f)}{n}.\end{aligned}$$

For $\Delta_n = \int_a^b h(t)dt - \frac{b-a}{n}\sum_{j=1}^n h(a + (b-a)j/n)$, we use the change of variables $\psi(x) = (b-a)x + a, x \in [0,1]$ and we obtain

$$\Delta_n = (b-a)\left\{ \int_0^1 (h \circ \psi)(x)dx - \frac{1}{n}\sum_{j=1}^n (h \circ \psi)(j/n) \right\}.$$

The previous results and Lemma F.6 (ii) yield

$$|\Delta_n| \leq (b-a)V_0^1(h \circ \psi)/n = (b-a)V_a^b(h)/n.$$

The second statement of the Lemma follows trivially from this. □

LEMMA F.11. *$W^{(T)}(x)$ is 2π periodic, $\int_{-\pi}^{\pi} W^{(T)}(x)\mathrm{d}x = 1$. Furthermore, if $B_T < 1$, $\|W^{(T)}\|_\infty = \frac{1}{B_T}\|W\|_\infty$, and we have $V_{-\pi}^{\pi}(W^{(T)}) = \frac{1}{B_T}V_{-\pi}^{\pi}(W)$.*

PROOF. Let us first prove that $\int_{-\pi}^{\pi} W^{(T)}(x)dx = 1$. Since

$$\int_{-\pi}^{\pi} W^{(T)}(x)dx = \sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{1}{B_T} W\left(\frac{x + 2\pi j}{B_T}\right) dx,$$

the change of variables $y = (x + 2\pi j)/B_T$ yields

$$\int_{-\pi}^{\pi} W^{(T)}(x)dx = \sum_{j\in\mathbb{Z}} \int_{(2j-1)\pi/B_T}^{(2j+1)\pi/B_T} W(y)dy = \int_{\mathbb{R}} W(y)dy = 1.$$

If $B_T < 1$, then for $x \in [-\pi, \pi], W^{(T)}(x) = \frac{1}{B_T} W(x/B_T)$. The third statement follows directly because of the periodicity, and the last statement follows from

$$V_{-\pi}^{\pi}(W^{(T)}) = V_{-\pi/B_T}^{\pi/B_T}(W/B_T) = \frac{1}{B_T} V_{-\pi}^{\pi}(W).$$

We have used Lemma F.6 and the fact that $W(x) = 0$ if $|x| \geq 1$. □

Using this lemma, we obtain

LEMMA F.12. *Provided $B_T \to 0$,*

$$\frac{2\pi}{T} \sum_{s=1}^{T-1} W^{(T)}(\omega - 2\pi s/T) = 1 + O(B_T^{-1} T^{-1}),$$

and the error is uniform in ω.

PROOF. Set

$$\begin{aligned} \Delta_n &= \int_{-\pi}^{\pi} W^{(T)}(\omega - \alpha) d\alpha - \frac{2\pi}{T} \sum_{s=1}^{T-1} W^{(T)}(\omega - 2\pi s/T), \\ &= \int_{-\pi}^{\pi} W'^{(T)}(\alpha) d\alpha - \frac{2\pi}{T} \sum_{s=1}^{T-1} W'^{(T)}(2\pi s/T), \end{aligned}$$

where $W'^{(T)}(\alpha) = W^{(T)}(\omega - \alpha)$. Lemmas F.6, F.10 and F.11 yield

$$|\Delta_n| \leq \frac{2\pi}{T}(V_{-\pi}^{\pi}(W'^{(T)}) + \|W'^{(T)}\|_\infty) = \frac{2\pi}{B_T T}(V_{-\pi}^{\pi}(W) + \|W\|_\infty),$$

which does not depend on ω, and is of order $O(B_T^{-1} T^{-1})$. Since

$$\int_{-\pi}^{\pi} W^{(T)}(\omega - \alpha) d\alpha = 1,$$

we obtain

$$\frac{2\pi}{T} \sum_{s=1}^{T-1} W^{(T)}(\omega - 2\pi s/T) = 1 + O(B_T^{-1} T^{-1}),$$

and the error is uniform in ω. □

LEMMA F.13. *Under condition **C(p,2)**, for each $k = 0, 1, \ldots, p$:*

$$\frac{\partial^k f_\omega}{\partial \omega^k} = \sum_{t \in \mathbb{Z}} (-\mathrm{i}t)^k e^{-\mathrm{i}\omega t} r_t,$$

and the convergence is L^2, uniformly ω. Moreover,

$$\sup_{\omega} \left\| \frac{\partial^k}{\partial \omega^k} f_\omega \right\|_2 < \infty \quad k = 1, 2, \ldots, p.$$

PROOF. Since $\mathbf{C(p,2)}$ implies $\mathbf{C(k,2)}$ for $k = 0, 1, \ldots, p$, the result follows by an iterative application of Rudin (1976, Theorem 7.17) to the projection of the partial sums

$$\sum_{t=-N}^{N} e^{-\mathrm{i}\omega t} r_t.$$

Hence

$$\sup_{\tau,\sigma,\omega} \left\| \frac{\partial^k}{\partial \omega^k} f_\omega \right\|_2 \leq \sum_t (1 + |t|^k) \|r_t\|_2 < \infty.$$

□

PROPOSITION F.14. *Let $a_s : [0,1]^2 \to \mathbb{R}; s = 0, \pm 1, \pm 2, \ldots$ be functions such that $\sum_{s \in \mathbb{Z}} \|a_s\|_2 < \infty$, and let A_s be the operator on $L^2([0,1], \mathbb{R})$ with kernel a_s. Let $\varepsilon_t, t = 0, \pm 1, \ldots$ be an iid sequence of random elements of $L^2([0,1], \mathbb{R})$ such that $\mathbb{E}\|\varepsilon_0\|_2^p < \infty, \quad p = 1, 2, \ldots$.*

Define $X_t^{(N)} = \sum_{s=-N}^{N} A_s \varepsilon_{t-s}$. Then, the linear process $X_t = \sum_{s \in \mathbb{Z}} A_s \varepsilon_{t-s}$ *has the following properties:*

1. *Norm-2 $L^p(\mathbb{P})$ convergence:* $\lim_{N \to \infty} \left(\mathbb{E} \left\| X_t^{(N)} - X_t \right\|_2^p \right)^{1/p} = 0$, *and* $\mathbb{E}\|X_t\|_2^p < \infty$.
2.
$$\|\mathrm{cum}\,(\varepsilon_0)_k\|_2 < \infty, \tag{F.2}$$

   where we have written $\mathrm{cum}\,(\varepsilon_0)_k = \mathrm{cum}\left(\underbrace{\varepsilon_0, \ldots, \varepsilon_0}_{k \text{ times}} \right)$.
3. *We have* $\mathrm{cum}\,(X_{t_1}, \ldots, X_{t_k}) = \sum_{s \in \mathbb{Z}} (A_{t_1 - s} \otimes \cdots \otimes A_{t_k - s}) \mathrm{cum}\,(\varepsilon_0)_k$, *where the convergence is in $\|\cdot\|_2$.*

PROOF. Let

$$X_t^{-(N)} = X_t - X_t^{(N)} = \sum_{|s| > N} A_s \varepsilon_{t-s},$$

for the tail of the series of X_t. Since

$$\|A_s \varepsilon_{t-s}\|_2 \leq |||A_s|||_\infty \|\varepsilon_{t-s}\|_2 = \|a_s\|_2 \|\varepsilon_{t-s}\|_2,$$

$$
\begin{aligned}
\mathbb{E}\left\|X_t^{-(N)}\right\|_2^p &\leq \sum_{|s_1|,\ldots,|s_p|>N} \|a_{s_1}\|_2 \cdots \|a_{s_p}\|_2 \mathbb{E}\left\{\|\varepsilon_{t-s_1}\|_2 \cdots \|\varepsilon_{t-s_p}\|_2\right\} \\
&\leq \sum_{|s_1|,\ldots,|s_p|>N} \|a_{s_1}\|_2 \cdots \|a_{s_p}\|_2 (\mathbb{E}\|\varepsilon_{t-s_1}\|_2^p)^{1/p} \cdots (\mathbb{E}\|\varepsilon_{t-s_p}\|_2^p)^{1/p} \\
&= \mathbb{E}\|\varepsilon_0\|_2^p \cdot \sum_{|s_1|,\ldots,|s_p|>N} \|a_{s_1}\|_2 \cdots \|a_{s_p}\|_2 \\
&= \mathbb{E}\|\varepsilon_0\|_2^p \cdot \left(\sum_{|s|>N} \|a_s\|_2\right)^p \\
&< \infty,
\end{aligned}
$$

where the inequality in the third line comes from the generalized Hölder inequality. Hence,

$$
\lim_{N\to\infty} \left(\mathbb{E}\left\|X_t^{-(N)}\right\|_2^p\right)^{1/p} = 0,
$$

norm-2 $L^p(\mathbb{P})$-convergence of $X_t^{(N)}$ to X_t.

Let us now show (F.2). For any partition $\nu = (\nu_1, \ldots, \nu_p)$ of $(1, \ldots, k)$, let $|\nu_j|$ denote the number of elements of the set ν_j, and $\int_{|\nu_j|}\int$ denote the integral over $[0,1]^{|\nu_j|}$. The triangle inequality and Jensen's inequality yield

$$
\begin{aligned}
\|\mathrm{cum}\,(\varepsilon_0)_k\|_p &\leq \sum_{\nu=(\nu_1,\ldots,\nu_q)} (q-1)! \prod_{l=1}^q \left\|\mathbb{E}\left[\prod_{j\in\nu_l} \varepsilon_0(\tau_j)\right]\right\|_p \\
&\leq \sum_{\nu=(\nu_1,\ldots,\nu_q)} (q-1)! \prod_{l=1}^q \mathbb{E}\left\|\prod_{j\in\nu_l} \varepsilon_0(\tau_j)\right\|_p \\
&= \sum_{\nu=(\nu_1,\ldots,\nu_q)} (q-1)! \prod_{l=1}^q \mathbb{E}\|\varepsilon_0\|_p^{|\nu_l|} \\
&< \infty.
\end{aligned}
$$

The sum is over all *unordered* partitions of $\{1, \ldots, k\}$. The last inequality follows naturally from the assumption $\mathbb{E}\|\varepsilon_0\|_p^k < \infty$, since $|\nu_l| \leq k$. Let us turn to the proof of the last statement of the proposition. We claim that

$$
\begin{aligned}
&\operatorname{cum}\left(X_{t_1}, \ldots, X_{t_k}\right) \\
&= \operatorname{cum}\left(\sum_{s_1 \in \mathbb{Z}} A_{s_1} \varepsilon_{t_1-s_1}, \ldots, \sum_{s_k \in \mathbb{Z}} A_{s_k} \varepsilon_{t_k-s_k}\right) \\
&= \sum_{s_1,\ldots,s_k \in \mathbb{Z}} \operatorname{cum}\left(A_{s_1} \varepsilon_{t_1-s_1}, \ldots, A_{s_k} \varepsilon_{t_k-s_k}\right) \qquad \text{(F.3)} \\
&= \sum_{s_1,\ldots,s_k \in \mathbb{Z}} (A_{s_1} \otimes \cdots \otimes A_{s_k}) \operatorname{cum}\left(\varepsilon_{t_1-s_1}, \ldots, \varepsilon_{t_k-s_k}\right) \\
&= \sum_{s \in \mathbb{Z}} (A_{t_1-s} \otimes \cdots \otimes A_{t_k-s}) \operatorname{cum}\left(\varepsilon_0\right)_k, \qquad \text{(F.4)}
\end{aligned}
$$

where the equality holds in $\|\cdot\|_2$; the first equality is the definition of X_t, and the last equality comes from the independence of the ε_t's. Since

$$\|(A_{t_1-s} \otimes \cdots \otimes A_{t_k-s}) \operatorname{cum}(\varepsilon_0)_k\|_2 \le \|a_{t_1-s}\|_2 \cdots \|a_{t_k-s}\|_2 \|\operatorname{cum}(\varepsilon_0)_k\|_2,$$

and

$$\sum_{s \in \mathbb{Z}} \|a_{t_1-s}\|_2 \cdots \|a_{t_k-s}\|_2 \le \sum_{s_1,\ldots,s_k \in \mathbb{Z}} \|a_{t_1-s_1}\|_2 \cdots \|a_{t_k-s_k}\|_2 = \left(\sum_{s \in \mathbb{Z}} \|a_s\|_2\right)^k,$$

the series (F.4) converges in $\|\cdot\|_2$.

We now justify the equality (F.3) and the one following it. For (F.3), recall the definition of the cumulant,

$$\operatorname{cum}\left(X_{t_1}, \ldots, X_{t_k}\right)(\tau_1, \ldots, \tau_k) = \sum_{\nu=(\nu_1,\ldots,\nu_q)} (-1)^{q-1}(q-1)! \prod_{l=1}^{q} \mathbb{E}\left[\prod_{j \in \nu_l} X_{t_j}(\tau_j)\right],$$

where the sum is over all *unordered* partitions of $\{1, \ldots, k\}$. First, notice that since $X_t^{(N)}$ converges in norm-2 $L^p(\mathbb{P})$-wise,

$$\mathbb{E}\left[\lim_{N\to\infty} X_{t_1}^{(N)} \otimes \cdots \otimes X_{t_l}^{(N)}\right] = \lim_{N\to\infty} \mathbb{E}\left[X_{t_1}^{(N)} \otimes \cdots \otimes X_{t_l}^{(N)}\right], \qquad \text{(F.5)}$$

in $\|\cdot\|_2$, where

$$X_{t_1}^{(N)} \otimes \cdots \otimes X_{t_l}^{(N)}(\tau_1, \ldots, \tau_l) = X_{t_1}^{(N)}(\tau_1) \cdots X_{t_l}^{(N)}(\tau_l).$$

Hence

$$
\begin{aligned}
&\operatorname{cum}\left(X_{t_1}, \ldots, X_{t_k}\right)\left(\tau_1, \ldots, \tau_k\right) \\
&= \sum_{\nu=(\nu_1,\ldots,\nu_q)} (-1)^{q-1}(q-1)! \prod_{l=1}^{q} \mathbb{E}\left[\prod_{j\in\nu_l} \lim_{N\to\infty} X_{t_j}^{(N)}(\tau_j)\right] \\
&= \sum_{\nu=(\nu_1,\ldots,\nu_q)} (-1)^{q-1}(q-1)! \prod_{l=1}^{q} \mathbb{E}\left[\lim_{N\to\infty} \prod_{j\in\nu_l} X_{t_j}^{(N)}(\tau_j)\right] \\
&= \lim_{N\to\infty} \sum_{\nu=(\nu_1,\ldots,\nu_q)} (-1)^{q-1}(q-1)! \prod_{l=1}^{q} \mathbb{E}\left[\prod_{j\in\nu_l} X_{t_j}^{(N)}(\tau_j)\right] \\
&= \lim_{N\to\infty} \operatorname{cum}\left(\sum_{|s_1|\le N} A_{s_1}\varepsilon_{t_1-s_1}, \ldots, \sum_{|s_k|\le N} A_{s_k}\varepsilon_{t_k-s_k}\right)(\tau_1,\ldots,\tau_k) \\
&= \lim_{N\to\infty} \sum_{|s_1|,\ldots,|s_k|\le N} \operatorname{cum}\left(A_{s_1}\varepsilon_{t_1-s_1}, \ldots, A_{s_k}\varepsilon_{t_k-s_k}\right)(\tau_1,\ldots,\tau_k) \\
&= \sum_{s_1,\ldots,s_k\in\mathbb{Z}} \operatorname{cum}\left(A_{s_1}\varepsilon_{t_1-s_1}, \ldots, A_{s_k}\varepsilon_{t_k-s_k}\right)(\tau_1,\ldots,\tau_k),
\end{aligned}
$$

where the equalities hold in $\|\cdot\|_2$, and the penultimate equality comes from the multilinearity of the cumulant.

Now that we have justified the equality preceding (F.3), let us justify the one following it. In the following, all integrals will be on hypercubes $[0,1]^i$ of the appropriate dimension.

$$
\begin{aligned}
&\operatorname{cum}\left(A_{s_1}\varepsilon_{t_1-s_1}, \ldots, A_{s_k}\varepsilon_{t_k-s_k}\right)(\tau_1,\ldots,\tau_k) \\
&= \sum_{\nu=(\nu_1,\ldots,\nu_q)} (-1)^{q-1}(q-1)! \prod_{l=1}^{q} \mathbb{E}\left[\prod_{j\in\nu_l} \int a_{s_j}(\tau_j,\sigma_j)\varepsilon_{t_j-s_j}(\sigma_j)d\sigma_j\right] \\
&= \sum_{\nu=(\nu_1,\ldots,\nu_q)} (-1)^{q-1}(q-1)! \prod_{l=1}^{q} \int_{|\nu_l|}\int \mathbb{E}\left[\prod_{j\in\nu_l} a_{s_j}(\tau_j,\sigma_j)\varepsilon_{t_j-s_j}(\sigma_j)\right] \prod_{j\in\nu_l} d\sigma_j \\
&= \int\cdots\int \sum_{\nu=(\nu_1,\ldots,\nu_q)} (-1)^{q-1}(q-1)! \prod_{l=1}^{q} \mathbb{E}\left[\prod_{j\in\nu_l} a_{s_j}(\tau_j,\sigma_j)\varepsilon_{t_j-s_j}(\sigma_j)\right] \prod_{j=1}^{k} d\sigma_j \\
&= \int\cdots\int \operatorname{cum}\left(a_{s_1}(\tau_1,\sigma_1)\varepsilon_{t_1-s_1}(\sigma_1), \ldots, a_{s_k}(\tau_k,\sigma_k)\varepsilon_{t_k-s_k}(\sigma_k)\right) \prod_{j=1}^{k} d\sigma_j
\end{aligned}
$$

$$= \int \cdots \int a_{s_1}(\tau_1, \sigma_1) \cdots a_{s_k}(\tau_k, \sigma_k) \mathrm{cum}\left(\varepsilon_{t_1 - s_1}(\sigma_1), \ldots, \varepsilon_{t_k - s_k}(\sigma_k)\right) \prod_{j=1}^{k} d\sigma_j$$
$$= \left[(A_{s_1} \otimes \cdots \otimes A_{s_k}) \mathrm{cum}\left(\varepsilon_{t_1 - s_1}, \ldots, \varepsilon_{t_k - s_k}\right)\right](\tau_1, \ldots, \tau_k),$$

where the equalities hold in $\|\cdot\|_2$. The only non-trivial equality is the second one, to which we now turn. Since we want to show an equality in $\|\cdot\|_2$, which is the norm of a Hilbert space $L^2([0,1]^i, \mathbb{R})$, we only need to show equality of the projections onto an orthonormal basis. Let $(\varphi_n)_{n=1,2,\ldots}$ be an orthonormal basis of $L^2([0,1], \mathbb{R})$. Then,

$$(\varphi_{n_1} \otimes \cdots \otimes \varphi_{n_k})_{n_1, \ldots, n_k = 1,2,\ldots}$$

is an orthonormal basis of $L^2([0,1]^k, \mathbb{R})$ and using the notation $\langle \cdot, \cdot \rangle$ for the scalar product, we get

$$\langle \mathbb{E}\left[A_{s_1}\varepsilon_{t_1 - s_1} \otimes \cdots A_{s_k}\varepsilon_{t_k - s_k}\right], \varphi_{n_1} \otimes \cdots \otimes \varphi_{n_k} \rangle$$
$$= \langle (A_{s_1} \otimes \cdots \otimes A_{s_k}) \mathbb{E}\left[\varepsilon_{t_1 - s_1} \otimes \cdots \otimes \varepsilon_{t_k - s_k}\right], \varphi_{n_1} \otimes \cdots \otimes \varphi_{n_k} \rangle,$$

which is justified by Tonelli's Theorem. Indeed,

$$\mathbb{E} \int \cdots \int \left|A_{s_1}(\tau_1, \sigma_1)\varphi_{n_1}(\tau_1)\varepsilon_{t_1 - s_1}(\sigma_1) \cdots A_{s_k}(\tau_k, \sigma_k)\varphi_{n_k}(\tau_k)\varepsilon_{t_k - s_k}(\sigma_k)\right| d\boldsymbol{\tau} d\boldsymbol{\sigma}$$
$$= \mathbb{E}\left[\iint \left|A_{s_1}(\tau, \sigma)\varphi_{n_1}(\tau)\varepsilon_{t_1 - s_1}(\sigma)\right| d\tau d\sigma \cdots \iint \left|A_{s_k}(\tau, \sigma)\varphi_{n_k}(\tau)\varepsilon_{t_k - s_k}(\sigma)\right| d\tau d\sigma\right]$$
$$\leq \|a_{s_1}\|_2 \cdots \|a_{s_k}\|_2 \mathbb{E}\left[\|\varepsilon_{t_1 - s_1}\| \cdots \|\varepsilon_{t_k - s_k}\|\right]$$
$$\leq \|a_{s_1}\|_2 \cdots \|a_{s_k}\|_2 \mathbb{E}\|\varepsilon_0\|_2^k$$
$$< \infty,$$

where the first inequality is justified by the Cauchy-Schwarz inequality, and the second one by the generalized Hölder inequality. This completes the proof. □

The following Lemma is a straightforward extension of results on approximate identities (see Edwards (1967, §3.2)) adapted to our framework:

LEMMA F.15 (Approximate identities). *Suppose $K_T, T = 1, 2, \ldots$ is a sequence of functions defined on $[-\pi, \pi]$ satisfying, as $T \to \infty$:*

(i) $\sup_T \int_{-\pi}^{\pi} |K_T(\alpha)| d\alpha < \infty$,
(ii) $\int_{-\pi}^{\pi} K_T(\alpha) d\alpha \longrightarrow 2\pi$

(iii) For all $\delta > 0$, $\int_{\delta \leq |\alpha| \leq \pi} |K_T(\alpha)| d\alpha \to 0$.

Let $E \subset \mathbb{R}$ *be an interval,*

$$g : [-\pi, \pi] \times E \to \mathbb{C}$$

be a function and, for each $e \in E$*, define* $g_e(\omega) = g(\omega, e)$*. Let* $g_\cdot(\omega)$ *denote the function* $e \mapsto g_e(\omega)$*.*

If the function $\omega \mapsto g_\cdot(\omega)$ *is uniformly continuous with respect to* $\|\cdot\|_p$*, meaning that* $\forall \varepsilon > 0, \exists \delta > 0$ *such that*

$$|\omega_1 - \omega_2| < \delta \implies \|g_\cdot(\omega_1) - g_\cdot(\omega_2)\|_p < \varepsilon, \tag{F.6}$$

and bounded with respect to $\|\cdot\|_p$*,*

$$\sup_\omega \|g_\cdot(\omega)\|_p < \infty,$$

then the convolution

$$K_T * g_e(\omega) = \int_{-\pi}^{\pi} K_T(\alpha) g_e(\omega - \alpha) d\alpha$$

converges in $\|\cdot\|_p$ *to* $g_e(\omega)$*, uniformly in* ω*:*

$$\sup_\omega \|K_T * g_\cdot(\omega) - g_\cdot(\omega)\|_p \to 0 \quad \text{as } T \to \infty.$$

Notice that if $p = \infty$, (F.6) *is the same as uniform equicontinuity of the family of functions* $\{g_e\}_{e \in E}$*.*

Proof. We follow the same strategy an in Edwards (1967, Theorem 3.2.2). We use the shorthand notation $\int = \int_{-\pi}^{\pi}$. Setting

$$a_T = \int K_T(\alpha) d\alpha,$$

we obtain via Jensen's inequality

$$\sup_\omega \|K_T * g_\cdot(\omega) - a_T g_\cdot(\omega)\|_p \leq \int |K_T(\alpha)| \cdot \sup_\omega \|g_\cdot(\omega - \alpha) - g_\cdot(\omega)\|_p d\alpha =: I.$$

Fix $\varepsilon > 0$. We can choose $\delta > 0$ satisfying (F.6). The rest of the proof parallels that of Edwards (1967); the idea is the following: we separate the integral $\int_i$ into

$$\int_{0 \leq |\alpha| \leq \delta} + \int_{\delta \leq |\alpha| \leq \pi}.$$

The first integral will be small because of the continuity condition and the boundedness of the L^1-norm of K_T, and the second integral is small of the property (iii) of an approximate identity, and because the $g_\cdot$'s are bounded. The proof then completed by noting that $a_T \to 2\pi$. □

REMARK F.16. *Notice that, in the previous Lemma, the rate of convergence depends only on*

1. *The properties of the approximate identity,*
2. *The bound* $\sup_\omega \|g_\cdot(\omega)\|_p$*,*
3. *The continuity parameter* $\delta = \delta(\varepsilon)$*.*

F.1. Some operator theory.

LEMMA F.17. *Let* $K \subset \mathbb{R}^n$ *be measurable and compact. Let* a, b, c *be continuous functions* $a, b, c : K \times K \to \mathbb{R}$*, with induced operators* A, B, C *on* $L^2(K, \mathbb{R})$*. That is,*

$$Af(\tau) = \int_K a(\tau, \sigma) f(\sigma) d\sigma \quad f \in L^2(K, \mathbb{R}).$$

We can define the product operator AB *of two operators by composition, i.e. the operator* AB *is defined by* $(AB)f = A(Bf)$ *for* $f \in L^2(K, \mathbb{R})$*, and has kernel*

$$r(\tau, \sigma) = \int_K a(\tau, \mu) b(\mu, \sigma) d\sigma.$$

This operation is associative.

We can also define the operator $(A \otimes B)$ *on* $L^2(K \times K, \mathbb{R})$ *acting as follows:*

$$[(A \otimes B)C](\tau, \sigma) = \iint_{K \times K} a(\tau, \mu_1) b(\sigma, \mu_2) c(\mu_1, \mu_2) d\mu_1 d\mu_2.$$

Since $(A \otimes B)C \in L^2(K \times K, \mathbb{R})$*, we can also view it as an operator on* $L^2(K, [0, 1])$*, and we have the equality* $(A \otimes B)C = ACB^\dagger$*, where* $B^\dagger$ *is the adjoint operator of* B*.*

PROOF. The fact that the product AB is a well defined operator follows easily from the continuity assumptions and the compactness of K. The associativity of the product is straightforward. Using Tonelli's Theorem, the formula for the kernel of AB follows directly. As for the equality $(A \otimes B)C = ACB^\dagger$, notice that the kernel of $B^\dagger$ is $r(\tau, \sigma) = b(\sigma, \tau)$, and an application of Tonelli's Theorem yields the result. □

F.1.1. *Schatten classes of Operators.* We recall here the definition of Schatten classes of operators (Ringrose (1971), Zhu (2007)), and some of their basic properties. Let H be a separable Hilbert space. If we denote by $\mathcal{L}_\infty(H)$ the space of bounded operators on H,we can define the Schatten p-class of H as follows:

Definition F.18. *For $1 \leq p < \infty$, the Schatten p-class of H is the subset $S_p \subset \mathcal{L}_\infty(H)$ consisting of all compact operators T for which*

$$\sum_{n \geq 1} |\langle Te_n, e_n \rangle|^p < \infty,$$

for all orthonormal bases $\{e_n\}$ of H; by convention $S_\infty = \mathcal{L}_\infty(H)$, the space of bounded operators.

It follows directly from the properties of ℓ_p spaces that $1 \leq p \leq q \leq \infty \implies S_p \subset S_q$. The following characterisation of the Schatten p-classes is a classical result:

Proposition F.19. *Let T be a compact operator on H, with singular value decomposition*

$$T = \sum_n \lambda_n \varphi_n \otimes \psi_n,$$

where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0.$$

For $1 \leq p < \infty$, we define the Schatten p-norm

$$|||T|||_p = \left[\sum_{n=1}^{\infty} \lambda_n^p \right]^{1/p}.$$

Then we have

$$T \in S_p \iff |||T|||_p < \infty,$$

for any orthonormal basis (e_n),

$$\sum_n |\langle Te_n, e_n \rangle|^p \leq |||T|||_p^p,$$

and the space $(S_p, |||\cdot|||_p)$ is a Banach space.

For $T \in \mathcal{L}_C(H)$, we define $|||T|||_\infty = \lambda_1$, and this coincides with the usual operator norm.

For any bounded operators A, B on H, we can define their product AB as being the bounded operator on H obtained by composition of A and B, $(AB)x = A(Bx), \quad x \in H$. We can now state a useful Hölder-type inequality for Schatten spaces:

Proposition F.20. *Let $1 \leq t, p, q \leq \infty$, such that $\frac{1}{t} = \frac{1}{p} + \frac{1}{q}$. If $A \in S_p, B \in S_q$, then $AB \in S_t$ and*

$$|||AB|||_t \leq |||A|||_p |||B|||_q.$$

Notice that the Schatten 1-norm is the nuclear norm. The Schatten 2-norm is also called the Hilbert-Schmidt norm, and we have the following property for $T \in S_2$:

$$|||T|||_2^2 = \sum_n \|Te_n\|^2,$$

where $e_1, e_2, \ldots$ is an orthonormal basis of H, and the sum is independent of the choice of the basis. It is also useful to recall the following result (Weidmann 1980, Theorem 6.11):

Proposition F.21. *Let $H = L^2(M, \mathbb{C})$, with M a measurable subset of $\mathbb{R}^n$. An operator $T : H \to H$ is Hilbert-Schmidt if and only if $\exists k \in L^2(M \times M, \mathbb{C})$ such that for all $f \in H$,*

$$Tf(x) = \int_M k(x,y) f(y) dy \qquad a.s.\ for\ x \in M.$$

We have $|||T|||_2 = \iint_{M\times M} |k(x,y)|^2 dxdy$, and the adjoint $T^\dagger$ is induced by the kernel $k^\dagger(x,y) = \overline{k(y,x)}$.

The following well-known result relates the expected squared norm of a random variable to the trace of its covariance operator:

Lemma F.22. *Let $K = \prod_{j=1}^n [a_j, b_j] \subset \mathbb{R}^n$, where $-\infty < a_j < b_j < \infty$ for $j = 1, \ldots, n$. Let μ denote Lebesgue measure on $\mathbb{R}^n$. Let X be a random element of*

$$L^2(K, \mathbb{C}) = \{f : K \to \mathbb{C} : \|f\|_2 < \infty\},$$

where $\|f\|_2 = \sqrt{\langle f, f\rangle}$ and $\langle f, g\rangle = \int_K f\bar{g} d\mu, \quad f, g \in L^2(K, \mathbb{C})$. Let $r(t,s) = \operatorname{cov}(X(t), X(s)), \quad t, s \in K$ be the covariance kernel of X, and $\mathscr{R}$ be the operator on $L^2(K, \mathbb{C})$ induced by the kernel. If $\mathbb{E}\|X\|_2^2 < \infty$, then $\mathscr{R}$ is trace-class and

$$\operatorname{trace}(\mathscr{R}) = \mathbb{E}\|X - \mathbb{E}X\|_2^2 = \int_K r(t,t) dt.$$

Proof. Since r is the covariance kernel of X,

$$r(t,s) = \overline{r(s,t)}, \quad \forall t, s \in K.$$

Moreover, $\mathscr{R}$ is a self-adjoint and non-negative operator. Since the space $L^2(K, \mathbb{C})$ is separable, for any orthonormal basis (e_n), we have

$$\operatorname{trace}(\mathscr{R}) = \sum_{n\geq 1} \langle \mathscr{R}e_n, e_n\rangle = \mathbb{E}\sum_{n\geq 1} |\langle X - \mathbb{E}X, e_n\rangle|^2 = \mathbb{E}\|X - \mathbb{E}X\|_2^2 < \infty,$$

thus $\mathscr{R}$ is trace-class. On the other hand, Tonelli's Theorem yields

$$\mathbb{E}\|X - \mathbb{E}X\|_2^2 = \int_K \mathbb{E}|X(t) - \mathbb{E}X(t)|^2 dt = \int_K r(t,t) dt.$$

Hence

$$\text{trace}\,(\mathscr{R}) = \int_K r(t,t) d\mu(t).$$

□

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